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ASYMPTOTIC SOLUTION OF THE PROBLEM OF HEAT TRANSFER BETWEEN TWO PLATES AND A UNIFORM FLUID FLOW

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For low Peclet numbers, analytic expressions are obtained for three highest terms of the asymptotic expansion of the solution of the convective heat-transfer problem for a system consisting of two parallel plates with identical lengths identical and constant surface temperatures and an infinite uniform fluid flow with a low Prandtl number.

Key words: *convective heat transfer, parallel plates, asymptotic solution, Peclet number.*

Introduction. The assumption that a fluid flow with a low Peclet number (Pe) is uniform implies that the Reynolds number is high and the Prandtl number is low. This case is seldom encountered in practice [1]. It occurs, however, in analyzing the filter-soil freezing process. As was shown in [2], the problem of the equilibrium shape of ice bodies formed around freezing columns in a plane filtration flow is solved in two stages. At the first stage, the Boussinesq transform for the convective heat-conduction equation in the thawed zone is used. As a result, in the complex hydrodynamic potential plane, there arises a heat-transfer problem for a system of parallel plates and a uniform fluid flow. At the second stage, a matching problem is to be solved.

A numerical analysis of the overall problem for two columns showed [3] that, in a certain range of the governing parameters (Pe number and column power Q), the problem has three solutions. Accordingly, the quasistationary process of increasing/decreasing the power Q with a fixed Pe displays a closing/opening hysteresis. To analytically substantiate the hysteresis phenomenon by asymptotic methods for the case $\mathrm{Pe} \ll 1$, it becomes necessary to take into account the third term of the asymptotics.

A three-term asymptotics of the solution of the problem of heat transfer between a single plate and a uniform fluid flow was obtained in [4], where an approach based on integral equations and the known spectral relation [5] were used. However, there is no such a spectral relation for two parallel plates. This is the reason for constructing the asymptotics by matching asymptotic expansions [6].

1. Formulation of the Problem. We assume that the plates of identical lengths with identical and constant surface temperatures are located symmetrically about the x and y axes, being oriented along the x axis (Fig. 1a). The incoming stream is also parallel to the x axis. In the dimensionless form, the steady-state convective heat transfer between such plates and an infinite fluid flow is described by the following system of equations [7]:

$$
\begin{gathered}
2\,\mathrm{Pe}\,\frac{\partial\Theta}{\partial x} = \Delta\Theta, \quad z \in D_z, \qquad \Theta = 0, \quad z \in \Gamma, \qquad \Theta = 1, \quad |z| \to \infty, \\
\frac{\partial\Theta}{\partial y} = 0, \quad y = 0, \qquad \Theta(x,-y) = \Theta(x,y), \quad z \in D_z.
\end{gathered}
\tag{1.1}
$$

Here D_z is the upper half of the physical plane for the variable $z = x + iy$ with the plate Γ cut off, $\Theta(x, y)$ is the dimensionless temperature distribution function, and Pe is the Peclet number calculated using half of the plate length as the length scale. The dashed line in Fig. 1a shows an auxiliary cut Λ formed by the part of the y axis above the plate. This cut will be further used to guarantee uniqueness of the auxiliary functions in the domain $D_z \cup \Lambda$ if their uniqueness in D_z is lacking. The point F is the origin. The points C, B, E, B', and C' lie on the

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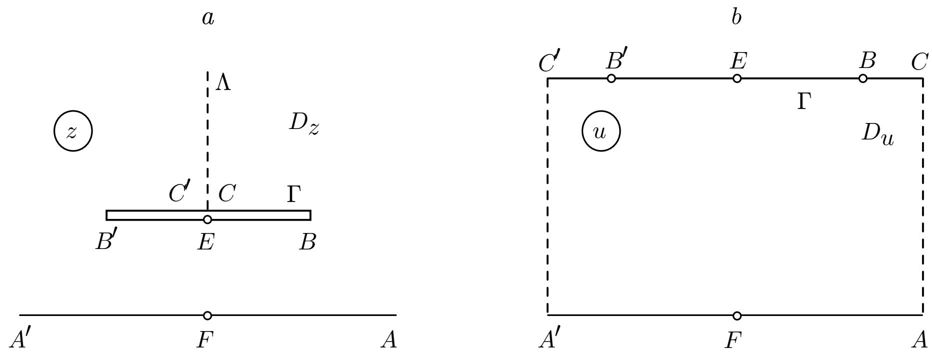

Fig. 1. Upper half of the physical plane z (a) and the plane of the auxiliary variable u (b).

plate and correspond to $z = ih$, $1 + ih$, ih, $-1 + ih$, and ih (h is half of the dimensionless distance between the plates). Although the points C, E, and C' correspond to one value of z, they should be distinguished: if we denote the upper and lower banks of the cut Γ as Γ^+ and Γ^- and the right and left banks of the cut Λ as Λ^+ and Λ^-, then we obtain $E \in \Gamma^-$, $C \in \Gamma^+ \cap \Lambda^+$, and $C' \in \Gamma^+ \cap \Lambda^-$. Formally, the last condition in system (1.1) is a redundant one for solving the problem in the domain D_z; nevertheless, it illustrates how this solution extends into the entire z plane.

By means of the transform $\Theta(x, y) = T(x, y) \exp(\mathrm{Pe}\, x)$, problem (1.1) can be reduced to the following problem for the function $T(x, y)$:

$$\begin{gathered}\Delta T = \mathrm{Pe}^2\, T, \quad z \in D_z, \qquad T = 0, \quad z \in \Gamma, \qquad T = \mathrm{e}^{-\mathrm{Pe}\, x}, \quad |z| \to \infty, \\ \frac{\partial T}{\partial y} = 0, \quad y = 0, \qquad T(x, -y) = T(x, y), \quad z \in D_z.\end{gathered} \tag{1.2}$$

If the boundary condition for Γ [2] is not satisfied, it is possible to represent the solution of this problem as the integral boundary operator

$$T(x, y) = \mathrm{e}^{-\mathrm{Pe}\, x} - \frac{1}{2\pi} \int\limits_{-1}^{1} \mu(x') \sum_{n=\pm 1} K_0\Big(\mathrm{Pe} \sqrt{(x - x')^2 + (y - nh)^2}\,\Big)\, dx', \tag{1.3}$$

where $\mu(x) = \partial T/\partial y\Big|_{\Gamma^+} - \partial T/\partial y\Big|_{\Gamma^-}$ is the density of the sources distributed along the contours Γ.

Formally, the integral boundary equation can be obtained using the boundary condition on the contour Γ from system (1.2). The solution of this equation ultimately defines the form of the function $T(x, y)$ and, hence, that of the function $\Theta(x, y)$. Accordingly, using the formulas

$$q = \int\limits_{-1}^{1} \mu(x)\, dx, \qquad Q = \int\limits_{-1}^{1} \mathrm{e}^{\mathrm{Pe}\, x}\, \mu(x)\, dx \tag{1.4}$$

we can find the total intensity of the sources (q) and the total heat flux to each plate (Q).

For an arbitrary Pe number, the integral boundary equation can be solved only numerically; however, for $\mathrm{Pe} = \varepsilon \ll 1$, it becomes possible to perform an asymptotic analysis of its solution. In this case, the density of the sources $\mu(x)$ and the function $T(x, y)$ can be represented as

$$\mu(x) = \mu_{,0}(x) + \varepsilon\mu_{,1}(x) + \varepsilon^2\mu_{,2}(x) + O(\varepsilon^3); \tag{1.5}$$

$$T(x, y) = T_{,0}(x, y) + \varepsilon T_{,1}(x, y) + \varepsilon^2 T_{,2}(x, y) + O(\varepsilon^3). \tag{1.6}$$

Here, the subscripts are also used to denote various constants and special functions; for this reason, in writing terms of the asymptotic expansion, we use commas placed before the subscripts.

Strictly speaking, the asymptotic expansions (1.5) and (1.6) should contain terms of the form $\varepsilon^\alpha(\ln \varepsilon)^\beta$ [5], i.e., the expansion coefficients $\mu_{,i}$ and $T_{,i}$ also depend on $\ln \varepsilon$. Nevertheless, this dependence is weak and does not change the asymptotic behavior of the expansions. For this reason, it is reasonable to use relations (1.5) and (1.6) to simplify subsequent manipulations.

In accordance with the matching method for asymptotic expansions [6], we seek solutions of two types: the solution in the vicinity of the plates and the solution at infinity; the lacking conditions for their complete determination will be sought by matching. It should be noted that the order of the differential equation changes in the vicinity of infinity. The corresponding asymptotic expansion is called the boundary-layer or internal expansion. To obtain it, we can use the generic form of solution (1.3) of a more general problem. The regular or external expansion can be obtained by formally decomposing problem (1.2) with respect to the small parameter ε for $|z| \sim 1$.

2. Internal (Boundary-Layer) Asymptotic Expansion. We choose the coordinates $X = \varepsilon x$ and $Y = \varepsilon y$ as the boundary-layer coordinates. We perform matching along the real axis; for this reason, only values of T for $Y = 0$ are needed. In view of this, from (1.3), we obtain

$$\pi(\mathrm{e}^{-X} - T\Big|_{Y=0}) = \int\limits_{-\varepsilon}^{\varepsilon} \mu\Big(\frac{X'}{\varepsilon}\Big) K_0\Big(\sqrt{(X - X')^2 + \varepsilon^2 h^2}\Big)\, \frac{dX'}{\varepsilon}. \tag{2.1}$$

We introduce three first moments for the integrand $\mu(x)$:

$$m_i = \int\limits_{-1}^{1} x^i \mu(x)\, dx, \qquad i = 0, 1, 2.$$

The following asymptotic expansion for the Bessel function K_0 in expression (2.1) is possible in the vicinity of $x \sim \infty$ such that $X \sim 1$, $X' \sim \varepsilon$, and $h \sim 1$:

$$K_0\Big(\sqrt{(X - X')^2 + \varepsilon^2 h^2}\Big) = K_0(X) + K_0'(X)(-X' + \varepsilon^2 h^2/(2X)) + K_0''(X) X'^2/2 + O(\varepsilon^3).$$

Substituting this expansion into formula (2.1), with accuracy to $O(\varepsilon^3)$, we find the expansion of the following form:

$$\mathrm{e}^{-X} - T\Big|_{Y=0} \approx \frac{m_0}{\pi} K_0\Big(X\Big) - \varepsilon \frac{m_1}{\pi} K_0'(X) + \frac{\varepsilon^2}{2\pi}\Big[m_0 h^2 \frac{K_0'(X)}{X} + m_2 K_0''(X)\Big]. \tag{2.2}$$

Note that the asymptotic representation (1.5) of the function $\mu(x)$ yields similar representations of m_i ($i = 0, 1, 2$):

$$m_0 = m_{0,0} + \varepsilon m_{0,1} + \varepsilon^2 m_{0,2} + O(\varepsilon^3), \qquad m_1 = m_{1,0} + \varepsilon m_{1,1} + O(\varepsilon^2), \qquad m_2 = m_{2,0} + O(\varepsilon).$$

Here, $m_{i,k}$ are yet indeterminate constants of the order of unity; it follows from expansion (2.2) that lower accuracy is sufficient to find higher-order moments.

3. Matching of the Regular Asymptotic and Boundary-Layer Expansions. Formally, the external (regular) expansion is expansion (1.6). After substituting it into problem (1.2) without the boundary condition at infinity, we obtain the problem

$$\begin{gathered} \Delta T_{,i} = 0, \quad z \in D_z, \qquad T_{,i} = 0, \quad z \in \Gamma, \\ \frac{\partial T_{,i}}{\partial y} = 0, \quad y = 0, \qquad T_{,i}(x, -y) = T_{,i}(x, y), \quad z \in D_z \end{gathered} \tag{3.1}$$

for $T_{,0}(x, y)$ and $T_{,1}(x, y)$ and the problem

$$\begin{gathered} \Delta T_{,2} = T_{,0}(x, y), \quad z \in D_z, \qquad T_{,2} = 0, \quad z \in \Gamma, \\ \frac{\partial T_{,2}}{\partial y} = 0, \quad y = 0, \qquad T_{,2}(x, -y) = T_{,2}(x, y), \quad z \in D_z \end{gathered} \tag{3.2}$$

for $T_{,0}(x, y)$. These problems are not closed since each of them contains no boundary condition at infinity. To derive these problems, we match the external expansion (1.6) with the internal one (2.2) along the real axis. This means that the function $T\Big|_{Y=0}$ defined by expression (2.2) for $X \to 0$ should have a structure similar to that of solution (1.6) of problems (3.1) and (3.2) for $y = 0$ and $x \to \infty$ [6]. It follows from the definition of the boundary-layer coordinates that it is possible to let X tend to zero as $x \to \infty$ by representing, e.g., X as $X \sim \sqrt{\varepsilon}$.

We substitute $X = \varepsilon x$ into expression (2.2) and then, using the asymptotic-expansion formulas for the function K_0 and its derivatives with respect to the small argument [8], we decompose the resultant expression with respect to small ε, retaining three highest terms of the asymptotics:

$$T\Big|_{\substack{Y=0,\\ X=\varepsilon x}} \approx 1 + \frac{m_0}{\pi}\Big[\rho(x)\Big(1+\varepsilon^2\,\frac{x^2}{4}\Big) - \varepsilon^2\,\frac{x^2}{4}\Big] - \varepsilon x + \varepsilon\,\frac{m_1}{\pi}\Big[\varepsilon\,\frac{x}{4} - \varepsilon\,\frac{\rho(x)x}{2} - \frac{1}{\varepsilon x}\Big]$$
$$+ \varepsilon^2\,\frac{x^2}{2} + \frac{\varepsilon^2}{2\pi}\Big\{m_0 h^2\Big[\frac{\rho(x)}{2} + \frac{1}{\varepsilon^2 x^2} - \frac{1}{4}\Big] + m_2\Big[\frac{\rho(x)}{2} - \frac{1}{\varepsilon^2 x^2} + \frac{1}{4}\Big]\Big\} + O(\varepsilon^3). \tag{3.3}$$

Here $\rho(x) = \ln(\varepsilon x/2) + \gamma$ and $\gamma \approx 0.53$ is the Euler constant. Separating terms of different orders in ε in (3.3) and using the notation $m_{i,k}$, we obtain the following conditions at infinity, $y = 0$ and $x \to \infty$, for $T_{,i}(x, y)$:

$$T_{,0}(x,y)\Big|_{y=0,\,x\to\infty} = 1 + \frac{m_{0,0}}{\pi}\,\rho(x) - \frac{m_{1,0}}{\pi}\,x^{-1} + \frac{m_{0,0}h^2 - m_{2,0}}{2\pi}\,x^{-2} + O(x^{-3}),$$
$$T_{,1}(x,y)\Big|_{y=0,\,x\to\infty} = -x + \frac{m_{0,1}}{\pi}\,\rho(x) - \frac{m_{1,1}}{\pi}\,x^{-1} + O(x^{-2}), \tag{3.4}$$
$$T_{,2}(x,y)\Big|_{y=0,\,x\to\infty} = x^2\Big[\frac{m_{0,0}}{4\pi}\,\rho(x) + \frac{1}{2} - \frac{m_{0,0}}{4\pi}\Big] + x\Big[\frac{m_{1,0}}{4\pi} - \frac{m_{1,0}}{2\pi}\,\rho(x)\Big]$$
$$+ \Big[\frac{4m_{0,2} + m_{0,0}h^2 + m_{2,0}}{4\pi}\,\rho(x) - \frac{m_{0,0}h^2 - m_{2,0}}{8\pi}\Big] + O(x^{-1}).$$

It should be noted that terms of orders ε and ε^2 in the boundary-layer expansion (2.2) contribute to the major term of the expansion with respect to ε [in expression (3.3), these are components that contain x^{-1} and x^{-2}, respectively]. The remainder term of expansion (2.2) also contributes to the major term of the regular expansion, which is emphasized by the symbol "≈" in relation (3.3). In addition, these terms are small in the expansion with respect to small $1/x$; for this reason, we use the sign of strict equality in relations (3.4); the corresponding contribution is included into the remainder terms.

The singular part of the first relation in (3.4) is the boundary-layer condition at infinity that closes problem (3.1) for $T_{,0}(x, y)$. This condition defines the structure of the function $T_{,0}(x, y)$; as a result, three regular terms of the relation (of the orders of unity, x^{-1}, and x^{-2}) yield three conditions for determining $m_{i,0}$.

Similarly, from the boundary condition at infinity for the function $T_{,1}(x, y)$ [the second relation in (3.4)], we can identify the singular part that, together with (3.1), defines the structure of the function $T_{,1}(x, y)$. Then, two regular terms of the expansion can be used to obtain two conditions for $m_{i,1}$. In a similar manner, we can find the structure of the function $T_{,2}(x, y)$.

4. Determination of the Major and Linear Terms of the Asymptotics. According to Sec. 3, the problem for the function $T_{,0}(x, y)$ can be written as system (3.1) with the singular part (3.4). Its solution is constructed using the method of singular points [9] with the use of elliptic functions; to represent these functions, an auxiliary plane $u = \xi + i\eta$ is needed. The domain D_z in the physical plane corresponds to the rectangle D_u in the u plane, shown in Fig. 1b. The points A' and A correspond to the values $u = 0$ and 1 and the points F and E correspond to the values $u = 1/2$ and $(1 + i\tau)/2$, respectively. The lateral sides of the rectangle D_u correspond to the auxiliary cut Λ in the z plane. Conformal mapping of the u plane onto the z plane is performed by the function [10]

$$z(u) = -\frac{h}{\pi}\,\frac{\vartheta_1'}{\vartheta_1}(u; i\tau), \tag{4.1}$$

where $\vartheta_1(u; i\tau)$ is the theta-function. The auxiliary parameter τ ($\operatorname{Im}\tau = 0$) is determined by the value of h only.

The analogy with the classical problem of electrical capacity of a plane capacitor allows us to argue that the dependence $h(\tau)$ is monotonic. A particular form of this dependence can be found as follows [10]. First, we consecutively find the values of α, s, and β:

$$\alpha(\tau) = \frac{\vartheta_2^2}{\vartheta_3^2}(0; i\tau), \qquad s(\tau) = \frac{1}{\alpha}\sqrt{\frac{K(\alpha^2) - E(\alpha^2)}{K(\alpha^2)}}, \qquad \beta(\tau) = \frac{F(\arcsin s; \alpha^2)}{2K(\alpha^2)}.$$

Here $K(\alpha^2)$ and $E(\alpha^2)$ are the elliptic integrals of the first and second kind; $F(\arcsin s; \alpha^2)$ is the incomplete elliptic integral of the first kind. The value of h can be calculated by the formula

$$h(\tau) = \pi(\vartheta_4/\vartheta_4')(\beta; i\tau). \tag{4.2}$$

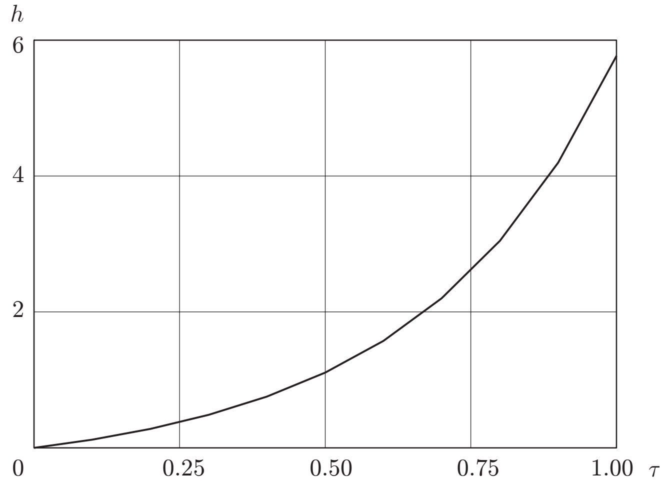

Fig. 2. The value of h as a function of the auxiliary parameter τ.

For $\tau \ll 1$, or, more exactly, for $\tau < 0.1$, these formulas cannot be used, because the value of α is close to unity and the calculations become insufficiently precise (for instance, when performed using the MAPLE packet). In the case considered, however, it suffices to estimate $h \approx \tau$, for instance, from physical considerations. Another numerical procedure applicable to the whole range of τ is given in [3]. The dependence $h(\tau)$ is shown in Fig. 2.

The harmonic function $T_{,0}(x, y)$ can be represented as

$$T_{,0}(x, y) = m_{0,0}[F(z) + F(\bar{z})]/(2\pi), \tag{4.3}$$

where $F(z)$ is an analytical function of the variable z such that $\operatorname{Im} F(z) = 0$ at $y = 0$. Using expression (4.1), we can pass from the variable z to the variable u. We denote the corresponding function as $f(u)$: $F[z(u)] = f(u)$. We also introduce the notation $F_u(z)$ and $F_{uu}(z)$: $F_u[z(u)] = f'(u)$ and $F_{uu}[z(u)] = f''(u)$.

4.1. *Construction and Analysis of the Function* $f(u)$. The infinity $x \to \infty$ in the physical plane corresponds to the point $u = 1$. We consider a vicinity of this point $u = 1 - \delta$, where $\delta \ll 1$. The quantity $\delta \ll 1$ can be related to $x \gg 1$ using formula (4.1) for the mapping $z(u)$. Using the Taylor expansion of the theta functions from (4.1), we obtain

$$x\Big|_{u=1-\delta,\,\delta\ll 1} = \frac{h}{\pi\delta}\Big[1 + \frac{\delta^2}{3}\,\frac{\vartheta_1'''}{\vartheta_1'}(0; i\tau)\Big] + O(\delta^3). \tag{4.4}$$

This formula can be inverted to give

$$\delta\Big|_{u=x\gg 1} = \frac{h}{\pi x}\Big[1 + \frac{1}{3}\Big(\frac{h}{\pi x}\Big)^2 \frac{\vartheta_1'''}{\vartheta_1'}(0; i\tau)\Big] + O(x^{-5}). \tag{4.5}$$

Passing to the function $f(u)$ in problem (3.1) with the singular part (3.4), with allowance for (4.4), we obtain the following boundary-value problem for the function $f'(u)$:

$$\begin{aligned} &\operatorname{Re} f'(u) \approx \delta^{-1}, \quad u = 1 - \delta, \qquad &&\operatorname{Re} f'(u) = 0, \quad u \in C'C, \\ &\operatorname{Im} f'(u) = 0, \quad u \in A'A, \qquad &&f'(u) = f'(u+1), \quad u \in A'C'. \end{aligned} \tag{4.6}$$

From the singularities, we find the elliptic function $f'(u)$ and then $f(u)$ [9]:

$$f'(u) = -\frac{d}{du}\,\ln\frac{\vartheta_1}{\vartheta_4}(u; 2i\tau), \qquad f(u) = -\ln\frac{\vartheta_1}{\vartheta_4}(u; 2i\tau). \tag{4.7}$$

Using the results obtained, we can describe in detail the behavior of the function $f(u)$ in the vicinity of the point $u = 1$. We obtain

$$\operatorname{Re} f(u)\Big|_{u=1-\delta} = -\ln\Big|\delta\,\frac{\vartheta_1'}{\vartheta_4}(0; 2i\tau)\Big| + \frac{\delta^2}{2}\Big[\frac{\vartheta_4'''}{\vartheta_4'}(0; 2i\tau) - \frac{1}{3}\,\frac{\vartheta_1'''}{\vartheta_1'}(0; 2i\tau)\Big] + O(\delta^4),$$

$$\operatorname{Re} f'(u)\Big|_{u=1-\delta} = \delta^{-1} - \delta\Big[\frac{\vartheta_4'''}{\vartheta_4'}(0; 2i\tau) - \frac{1}{3}\,\frac{\vartheta_1'''}{\vartheta_1'}(0; 2i\tau)\Big] + O(\delta^3),$$

$$\operatorname{Re} f''(u)\Big|_{u=1-\delta} = \delta^{-2} + \Big[\frac{\vartheta_4'''}{\vartheta_4'}(0; 2i\tau) - \frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'}(0; 2i\tau)\Big] + O(\delta^2).$$

Using the dependence $\delta(x)$ (4.5) in these relations, we examine the behavior of the functions $F(z)$, $F_u(z)$, and $F_{uu}(z)$ at infinity, $z = x \to \infty$:

$$\operatorname{Re} F(z) = \ln x + C_0 + C_2 x^{-2} + O(x^{-4}),$$

$$\operatorname{Re} F_u(z) = (\pi/h)(x - C_1/x) + O(x^{-3}), \qquad \operatorname{Re} F_{uu}(z) = (\pi^2/h^2)(x^2 + 2C_2) + O(x^{-2}). \tag{4.8}$$

Here C_i is a function of the parameter τ:

$$C_0 = \ln\Big|\frac{\pi}{h(\tau)}\frac{\vartheta_4}{\vartheta_1'}(0; 2i\tau)\Big|, \qquad C_1 = \frac{h^2(\tau)}{3\pi^2}\sum_{k=2}^{4}\Big[a_k\frac{\vartheta_k''}{\vartheta_k}(0; 2i\tau) + \frac{\vartheta_k''}{\vartheta_k}(0; i\tau)\Big],$$

$$C_2 = \frac{h^2(\tau)}{3\pi^2}\sum_{k=2}^{4}\Big[\frac{a_k}{2}\frac{\vartheta_k''}{\vartheta_k}(0; 2i\tau) - \frac{\vartheta_k''}{\vartheta_k}(0; i\tau)\Big], \qquad a_2 = a_3 = -1, \quad a_4 = 2.$$

Finally, let us write out the problem for the function $f''(u)$. It follows from the system of Eqs. (4.6) for $f'(u)$ that

$$\operatorname{Re} f''(u) \approx \delta^{-2}, \quad u = 1 - \delta, \qquad \operatorname{Re} f''(u) = 0, \quad u \in C'C,$$

$$\operatorname{Im} f''(u) = 0, \quad u \in A'A, \qquad f''(u) = f''(u+1), \quad u \in A'C'.$$

4.2. *Construction of the Functions $T_{,0}(x, y)$ and $T_{,1}(x, y)$.* Formulas (4.1), (4.3), and (4.7) determine the form of the function $T_{,0}(x, y)$. From the first relation of (4.8), we can find the behavior of the function $T_{,0}(x, y)$ at infinity $z = x \to \infty$:

$$T_{,0}(x, y)\Big|_{y=0,\, x\to\infty} = m_{0,0}(\ln x + C_0 + C_2 x^{-2})/\pi + O(x^{-4}).$$

Comparing this expression with (3.4), we find the relation of $m_{i,0}$ $(i = 1, 2, 3)$ with ε and τ:

$$\pi/m_{0,0} = \ln(2/\varepsilon) - \gamma + C_0, \qquad m_{1,0} = 0, \qquad m_{2,0} = m_{0,0}(h^2 - 2C_2). \tag{4.9}$$

The problem for $T_{,1}(x, y)$ can be written as system (3.1) and a condition at infinity given by the singular part of the second relation in (3.4). Note that the second singular component in the condition at infinity has the same structure as the singular part of condition (3.4) for $T_{,0}(x, y)$. For this reason, there appear the component $m_{0,1}\pi^{-1}f(u)$ in the function $T_{,1}(x, y)$ and the component of the form $m_{0,1}\pi^{-1}C_0$ in the term of the order of unity in the expansion of $T_{,1}(x, y)$ at infinity. Simultaneously, the first singular component in the condition at infinity provides no contribution of the order of unity to this expansion (see below). As a result, the regular term of the second relation in (3.4) of the order of unity gives a condition of the form $m_{0,1}[\ln(\varepsilon/2) + \gamma] = m_{0,1}C_0$, which, by virtue of the first relation in (4.9), is satisfied only if $m_{0,1} = 0$. Thus, there remains only the first component in the singular part of the condition at infinity for $T_{,1}(x, y)$.

Comparing the problem for $T_{,1}(x, y)$ with problem (4.6) for the function $f'(u)$, we obtain

$$T_{,1}(x, y) = -(h/\pi)\operatorname{Re} F_u(z).$$

Then, the second expression in (4.8) yields the expansion of $T_{,1}(x, y)$ at infinity:

$$T_{,1}(x, y)\Big|_{y=0,\, x\to\infty} = -x + C_1 x^{-1} + O(x^{-3}). \tag{4.10}$$

Comparing the terms of order x^{-1} in relation (4.10) and the second relation of (3.4), we obtain $m_{1,1} = -\pi C_1$.

Note, similarly to the case of a single plate [4], the function $T_{,0}$ has even symmetry and the function $T_{,1}$ has odd symmetry with respect to the y axis.

5. Determination of the Third Term of the Asymptotics. We can simplify the boundary condition for the function $T_{,2}$ at infinity [the third relation of (3.4)] using the expressions found previously for $m_{i,k}$. With accuracy to $O(x^{-1})$, we obtain

$$\frac{\pi}{m_{0,0}}T_{,2}(x, y) \approx \frac{x^2\ln x}{4} + \Big(C_0 - 1 - \frac{\pi}{m_{0,0}}\Big)\frac{x^2}{4} + \Big(\frac{m_{0,2}}{m_{0,0}} + \frac{h^2 - C_2}{2}\Big)\Big(\ln\frac{\varepsilon x}{2} + \gamma\Big) - \frac{C_2}{4}. \tag{5.1}$$

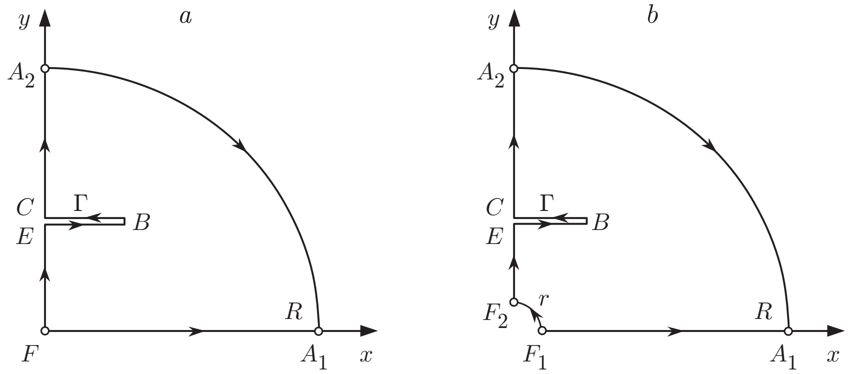

Fig. 3. Integration contours in the z plane used to calculate P_∞ (a) and G_∞ (b).

As in Sec. 4, the structure of the function $T_{,2}(x, y)$ as a solution of system (3.2), (5.1) is determined by singular terms, namely, by the first three components in braces in condition (5.1). In this case, however, the function $T_{,2}(x, y)$ is biharmonic and displays a more complex behavior at infinity.

We introduce the auxiliary functions $P(z)$ and $G(z)$

$$P(z) = \int_0^z F(z')\,dz', \qquad G(z) = \frac{1}{2}\int_{z_E}^z P(z')\,dz' \tag{5.2}$$

and analyze their behavior.

5.1. *Analysis of the Auxiliary Functions.* The function $T(x, y)$ is directly related to the temperature distribution; therefore, it is uniquely defined, together with its derivatives, everywhere in the domain D_z. Accordingly, the functions $T_{,i}(x, y)$ $(i = 1, 2, 3)$ and their derivatives are also uniquely defined. Meanwhile, the function $F(z)$ is nonunique in D_z: on passing around the cut Γ along each closed contour, the imaginary part of this function increases by π. If we draw an auxiliary cut Λ (see Fig. 1a), on which the function $\operatorname{Im} F(z)$ has a jump, then the function $F(z)$ becomes uniquely defined in the domain $D_z \cup \Lambda$. It follows from the definition of the functions $P(z)$ and $G(z)$ that they are nonunique in the domain D_z.

Based on the above considerations, it can be argued that the function $F(z)$ in the domain $D_z \cup \Lambda$ satisfies the boundary conditions

$$\operatorname{Re} F(z) = 0, \quad z \in \Gamma, \qquad \operatorname{Im} F(z) = 0, \quad y = 0, \qquad \operatorname{Im} F(z) = \pm H(y - h)\pi/2, \quad x = \pm 0$$

(H is the Heaviside function and $x = \pm 0$ is the approaching to the y axis from the right and from the left) and the condition at infinity $|z| \to \infty$

$$\operatorname{Re}\ F(z)|_{|z|\to\infty} = \ln z + C_0 + C_2 z^{-2} + O(|z|^{-4}),$$

which follows from (4.8) by virtue of the fact that $0 \leqslant \arg z \leqslant \pi/2$ and the major branch of the logarithm is used.

As follows from the definition of the function $P(z)$, it satisfies, in the domain $D_z \cup \Lambda$, the boundary conditions

$$\begin{gathered} \operatorname{Re} P(z) = 0, \quad z \in \Gamma, \qquad \operatorname{Im} P(z) = 0, \quad y = 0, \\ \operatorname{Re} P(z) = \mp (y - h) H(y - h)\pi/2, \qquad x = \pm 0 \end{gathered} \tag{5.3}$$

and the condition at infinity $|z| \to \infty$

$$P(z)\Big|_{|z|\to\infty} = z \ln z + (C_0 - 1)z + P_\infty - C_2 z^{-1} + O(z^{-3}). \tag{5.4}$$

Here P_∞ is the term of the expansion of the order of unity. Let us find its form.

In the z plane, we draw a circumference of a large radius $R \gg 1$ whose center lies at the origin (Fig. 3a). We denote the points of its interaction with the coordinate axes as A_1 and A_2. From the definition of the function $P(z)$, we have

$$\operatorname{Re} P(z)\Big|_{z=R} = \operatorname{Re} \int_{FA_1} F(z')\,dz'.$$

Applying the Cauchy theorem [11] to the analytical function $F(z)$, we change the integration path FA_1 to $FEBCA_2A_1$. As a result, we obtain the relation

$$\operatorname{Re} P(z)\Big|_{z=R} = R\ln R + (C_0-1)R + \pi h/2 - C_2/R + O(R^{-3}).$$

Comparing this relation with (5.4), we obtain the expression $P_\infty(\tau) = \pi h/2$. Here and below, in a similar situation for $G(z)$, we use the condition at infinity $|z| \to \infty$ and not only $z = x \to \infty$.

In compliance with (5.3) and (5.4) and the definition of the function $G(z)$, the latter satisfies, in the domain $D_z \cup \Lambda$, the boundary conditions

$$\begin{gathered}\operatorname{Re} G(z) = 0, \quad z \in \Gamma, \qquad \operatorname{Im} G(z) = 0, \quad y = 0, \\ \operatorname{Im} G(z) = \pm[h^2 - 2C_2 - (y-h)^2]H(y-h)\pi/8, \qquad x = \pm 0\end{gathered} \tag{5.5}$$

and the condition at infinity $|z| \to \infty$

$$G(z)\Big|_{|z|\to\infty} = \frac{z^2}{4}\ln z + \Big(C_0 - \frac{3}{2}\Big)\frac{z^2}{4} + \frac{\pi h}{4}\,z - \frac{C_2}{2}\,\ln z + G_\infty + O(|z|^{-2}). \tag{5.6}$$

In deriving the last condition in (5.5), we used the value of $\operatorname{Im} G(z_C) = (h^2 - 2C_2)\pi/8$ calculated using the last expression in (4.9). In (5.6), G_∞ is the expansion term of the order of unity, which is ultimately determined by formulas (5.2) and the known form of the function $F(z)$. Obviously, G_∞ depends on τ only. Let us find this dependence.

In the z plane, we draw circumferences of large radius $R \gg 1$ and small radius $r \ll 1$ whose centers coincide with the origin. We denote the points of intersection of the first circumference with the coordinate axes x and y as A_1 and A_2, and those of the second circumference as F_1 and F_2, respectively (Fig. 3b). Applying the Cauchy theorem [11] to the analytical function $G(z)/z$ and integration contour $\Sigma = A_1F_1F_2EBCA_2A_1$, we obtain

$$\oint\limits_\Sigma \frac{G(z)}{z}\,dz = 0.$$

After integration and subsequent transition to the limit $R \to \infty$, $r \to 0$, we obtain

$$G_\infty = \frac{J_1}{2} - \frac{J_2}{2\pi} - \frac{h^2}{2}, \qquad J_1 = \int\limits_{u_F}^{u_E} f'(u)\Big(\frac{y^2}{2} - hy\Big)_{y=\operatorname{Im} z(u)}\,du,$$

$$J_2 = \int\limits_{u_E}^{u_C} \operatorname{Im} f'(u)\Big[\frac{x^2-h^2}{2}\ln(x^2+h^2) - \frac{3x^2+h^2}{2} + 2hx\arctan\frac{x}{h}\Big]_{x=\operatorname{Re} z(u)}\,du.$$

Here J_1 and J_2 are real-valued quantities.

By virtue of the Cauchy–Riemann relations [11], the jump of the function $G(z)$ of the form (5.5) across the cut Λ results in the jump of the derivative of the function $\operatorname{Re} G(z)$ in the x direction: $\partial \operatorname{Re} G(z)/\partial x\Big|_{\Lambda^\pm} = \mp(y-h)\pi/4$.

5.2. *Construction of the Function* $T_{,2}(x,y)$. We seek the function $T_{,2}(x,y)$ in the form

$$\pi T_{,2}(x,y)/m_{0,0} = (z+\bar z)[P(z) + P(\bar z)]/8 - \operatorname{Re} G(z) + \operatorname{Re}\Omega(z), \tag{5.7}$$

where $\Omega(z)$ is a function unknown at the moment. Applying the Laplace operator to expression (5.7), we find that the Poisson equation in system (3.2) is satisfied by virtue of the first term in (5.7). The second term guarantees the required form of the highest singular term of the expansion of $T_{,2}(x,y)$ at infinity (5.1) — the term of order $x^2\ln x$; correspondingly, it compensates for the jumps of the derivative of the first term in (5.7) in the x direction across the cut Λ.

We write the problem for the analytical function $\Omega(z)$. From system (3.2) and expressions (5.3) and (5.5), we find the boundary conditions

$$\operatorname{Re}\Omega(z) = 0, \quad z \in \Gamma, \qquad \operatorname{Im}\Omega(z) = 0, \quad y = 0, \qquad \operatorname{Im}\Omega(z) = 0, \quad x = 0,$$

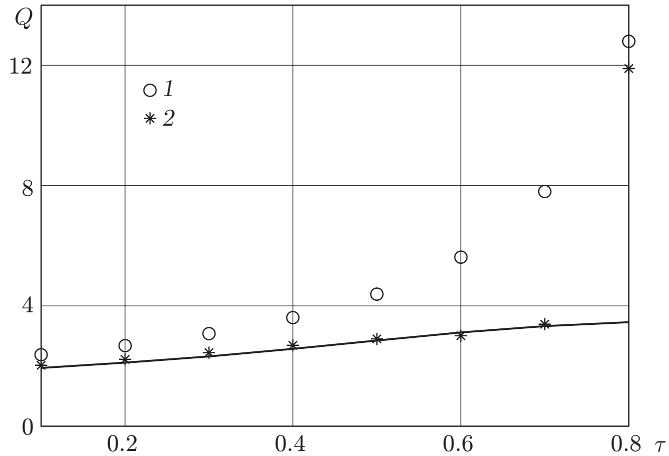

Fig. 4. Comparison of the asymptotic dependence $Q(\tau)$ with the numerical data obtained by the calculation procedure of [3] for Pe = 0.5: the curve show the numerical calculations by the procedure of [3]; points 1 and 2 refer to the calculations by the asymptotic formula (6.1) in the zero-order approximation and in the second-order approximation, respectively.

and from formulas (5.1), (5.4), and (5.6), we find the condition for $z = x \to \infty$

$$\operatorname{Re}\Omega(z) = \Big(\frac{2\pi}{m_{0,0}} - 1\Big)\frac{x^2}{8} + \frac{2m_{0,2} + m_{2,0}}{2m_{0,0}}\Big(\ln\frac{\varepsilon x}{2} + \gamma\Big) + \frac{C_2}{2}\Big(\ln\frac{\varepsilon}{2} + \gamma + \frac{1}{2}\Big) + G_\infty, \tag{5.8}$$

where, for simplicity, we used relations (4.9). A comparison of this problem with the problem for the functions $F(z)$ and $F_{uu}(z)$ show that the function $\Omega(z)$ can be represented as a linear combination of these functions:

$$\Omega(z) = \Big(\frac{2\pi}{m_{0,0}} - 1\Big)\frac{h^2}{8\pi^2}\,F_{uu}(z) + \frac{2m_{0,2} + m_{2,0}}{2m_{0,0}}\,F(z).$$

By virtue of conditions (4.8), the expansion $\operatorname{Re}\Omega(z)$ at infinity has the form

$$\operatorname{Re}\Omega(z)\Big|_{z=x\to\infty} = \Big(\frac{2\pi}{m_{0,0}} - 1\Big)\Big[\frac{x^2}{8} + \frac{C_2}{4}\Big] + \frac{2m_{0,2} + m_{2,0}}{2m_{0,0}}\,[\ln x + C_0] + O(x^{-2}).$$

Comparing the latter expression with (5.8), we obtain the following relation between $m_{0,2}$ and $G_\infty(\tau)$:

$$\Big(\frac{2m_{0,2} + m_{2,0}}{2m_{0,0}} + \frac{C_2}{2}\Big)\Big(\ln\frac{\varepsilon}{2} + \gamma\Big) + G_\infty = \Big(\frac{\pi}{m_{0,0}} - 1\Big)\frac{C_2}{2} + \frac{2m_{0,2} + m_{2,0}}{2m_{0,0}}\,C_0.$$

Finally, using (4.9), we find the last unknown moment of the function $\mu(x)$:

$$m_{0,2} = (2G_\infty + C_2(1 + C_0))m_{0,0}^2/(2\pi) - h^2 m_{0,0}/2.$$

6. Analysis of Results. We give the final form of the terms of the asymptotic expansion (1.6) of the function $T(x, y)$ in terms of the analytical functions $z(u)$ and $f(u)$ and moments $m_{i,j}$:

$$T_{,0}(x,y) = m_{0,0}\,\frac{\operatorname{Re} f(u)}{\pi}, \quad T_{,1}(x,y) = -h\,\frac{\operatorname{Re} f'(u)}{\pi}, \quad T_{,2}(x,y) = m_{0,0}\,\frac{\operatorname{Re} w(u) - J_1}{2\pi}.$$

Here

$$w(u) = f(u)\Big[|z^2(u)| + \frac{z^2(u)}{2}\Big] - \int\limits_{1/2}^{u}\Big[\overline{z(u)}\,z(\zeta) + \frac{z^2(\zeta)}{2}\Big]f'(\zeta)\,d\zeta + \Big(\frac{2\pi}{m_{0,0}} - 1\Big)\frac{h^2}{4\pi^2}\,f''(u) + \frac{2m_{0,2} + m_{2,0}}{m_{0,0}}\,f(u).$$

Using formula (1.4), we can calculate the total heat flux to the plate Q as a function of Pe and τ. With only nonzero moments $m_{i,j}$ retained, we have

$$Q = m_{0,0} + \varepsilon^2(m_{1,1} + m_{0,2} + m_{2,0}/2). \tag{6.1}$$

To pass in this formula from the parameter τ to the parameter h, we can use relation (4.2).

To check the validity of the formulas obtained for $\mathrm{Pe} = 0.5$, we compared the data obtained by the asymptotic dependence (6.1) with the dependence obtained by solving the integral boundary equation (1.4) numerically, using the procedure of [3]. The results are plotted in Fig. 4. Although the Peclet number here is not very low here, the allowance for the second term of the asymptotics appreciably improves the agreement with the calculations of [3] up to $\tau = 0.7$. The difference in the values of Q at large τ can be explained by the increase in h. In obtaining the asymptotics, we assumed that $h \sim 1$; in accordance with Fig. 2, we have $h \approx 3$ for $\tau = 0.8$. A similar comparison for $\mathrm{Pe} = 0.1$ yields almost a complete coincidence of the asymptotic dependence $Q(\tau)$ and the dependence calculated by the procedure of [3] up to $\tau = 1$ (for greater values of τ, no calculations were performed).

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